Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

NON-ESSENTIAL SINGULARITIES OF FUNCTIONS OF SEVERAL COMPLEX VARIABLES.

BY DUNHAM JACKSON.

It is a familiar property of functions of a single complex variable that if $f(z)$ has a pole at the point a, it can be expressed throughout the neighborhood of this point in the form $f(z) = \varphi(z)/(z - a)^m$, where m is a suitable positive integer and $\varphi(z)$ is a function of z which is analytic at a and does not vanish there. Osgood has recently extended this theorem, with appropriate modifications, to functions of several variables. An extension with somewhat different hypotheses, obtained simultaneously and independently by the present writer, is given below. It is assumed that $f(x_1, x_2, \cdots, x_n)$ is analytic throughout the neighborhood of a point, which for convenience is taken as the origin, except for singularities which satisfy a certain requirement of continuity in their distribution,* and which are of such a nature that f is analytic except for poles when regarded as a function of x_1 alone, and it is shown that under these conditions, f can be expressed as the quotient of two functions, each analytic in all the variables in the neighborhood of the origin. The denominator is of course no longer merely a power of a linear factor, in general, and the numerator may vanish at points of the neighborhood in question. The method used is essentially the same as one which is employed in well-known presentations of the proof of Weierstrass's theorem of factorization. The results are expressed in three theorems, the scope of each of which is indicated by the corresponding section-heading. No attempt has been made to obtain the greatest possible generality in the hypotheses, and further generalization is undoubtedly practicable; the aim has been to bring out clearly the central idea of the argument, without unnecessary complications.

1. Non-essential singularity of the first kind. Let the function $f(x_1, x_2, \cdots, x_n)$ be so defined in a region containing the point $(a_1, a_2, \cdots, a_n)$, that if fixed values are assigned to the last $n - 1$ variables, the resulting function of x_1 is analytic except for poles.† The distribution of the poles will be said to be *continuous* at the point $(a_1, a_2, \cdots, a_n)$, if the following condition is fulfilled: There exists a positive number r_1, as small as may be desired, with a corresponding positive number r_2, such that when the inequalities $|x_i - a_i| \leqq r_2$, $i = 2, \cdots, n$, are satisfied, there are no poles on the boundary of the circle $|x_1 - a_1| \leqq r_1$, and the sum of the orders of the poles inside the circle is constant. It will be observed that if this condition is satisfied, and if $f(x_1, a_2, \cdots, a_n)$ has no pole for $x_1 = a_1$, there is no pole at all in the neighborhood of $(a_1, \cdots, a_n)$. Furthermore, an application of the Heine-Borel theorem shows that if E is a closed set of points in the x_1-plane in which $f(x_1, a_2, \cdots, a_n)$ has no pole, and the distribution of poles is continuous at every point of E when $x_2 = a_2, \cdots, x_n = a_n$, then there exists a positive number ρ such that $f(x_1, x_2, \cdots, x_n)$ has no pole in E when $|x_i - a_i| \leqq \rho$, $i = 2, 3, \cdots, n$.

---

* In Osgood's theorem, the singularities are supposed to lie on analytic manifolds.

† Wherever the word *pole* is used in this paper, it will be understood to refer to the behavior of the function under consideration as a function of a single complex variable, when particular values are assigned to such other variables as may be involved. For example, the statement

The point about which the discussion centers will be taken for convenience as the point $(0, 0, \cdots, 0)$.

THEOREM I. *Let* $f(x_1, x_2, \cdots, x_n)$ *be a function of the n complex variables* $x_1, x_2, \cdots, x_n$, *having the following properties in the neighborhood of the origin:*

(*a*) *If* $x_2, \cdots, x_n$, *are held fast,* $f(x_1, x_2, \cdots, x_n)$ *is an analytic function of* x_1, *except for poles.*

(*b*) *The distribution of these poles is continuous at the origin.*

(*c*) *Except at the points corresponding to poles, f is analytic in all n variables together.*

Then f can be represented throughout a sufficiently restricted neighborhood of the origin, except at the singular points specified above, in the form

$$f(x_1, x_2, \cdots, x_n) = \frac{\varphi(x_1, x_2, \cdots, x_n)}{\psi(x_1, x_2, \cdots, x_n)},$$

where φ *and* ψ *are analytic in all n variables throughout the neighborhood in question, and* φ *does not vanish there.*

The denominator, here and in the later theorems, may be taken as a polynomial in the first variable with coefficients which are analytic functions of the remaining variables.

It is obvious that there is nothing to be proved unless $f(x_1, 0, \cdots, 0)$ has a pole at the point $x_1 = 0$. Let it be assumed, then, that such a pole is present, and that its order is m. If r_1 is a sufficiently small positive quantity, there will be no other pole of the function $f(x_1, 0, \cdots, 0)$, and

---

that $f(x_1, x_2, \cdots, x_n)$ has no poles in a certain region of the space of the n variables will mean that if $(a_1, a_2, \cdots, a_n)$ is any point of the region, the function $f(x_1, a_2, \cdots, a_n)$ has no pole for $x_1 = a_1$. A special *name* for a singular point of the kind indicated by the title of this section, at which the function becomes infinite with regard to all its arguments together, will not be needed again.

no zero of this function at all, inside or on the circumference $|x_1| = r_1$. In consequence of the hypothesis (*b*), it is possible to associate with a number r_1 satisfying this condition, a second positive number r_2, so that if

$$(1) \qquad |x_i| \leqq r_2, \qquad i = 2, \cdots, n,$$

the function $f(x_1, x_2, \cdots, x_n)$, considered as to its dependence on x_1 alone, has no pole for $|x_1| = r_1$, and has just m poles for $|x_1| < r_1$, when multiplicities are taken into account; and it may be assumed further that there are no zeros on the circumference $|x_1| = r_1$. For the last statement is surely correct if r_2 is sufficiently small, since f is continuous in all n variables and different from zero for $|x_1| = r_1, x_2 = \cdots = x_n = 0$; and a value of r_2 once chosen to satisfy the earlier requirements may be replaced by any smaller value.

It appears that the function f_{x_1}/f is analytic in all n variables for $|x_1| = r_1$, if the last $n - 1$ variables are subject to the restriction (1), as will be assumed henceforth. The integral of this quotient, extended around the circumference, is an analytic function of $x_2, \cdots, x_n$, and retains this character if the integrand is replaced by $x_1^k f_{x_1}/f$, where k is any positive integer. Let us consider first the integral of f_{x_1}/f itself. Except for a factor $2\pi i$, its value is the difference between the number of poles and the number of zeros of f inside the circle. As this difference is an integer, the statement that it is an analytic, and hence continuous, function of $x_2, \cdots, x_n$, is equivalent to the statement that it is constant. But the number of poles has been assumed to be constant, by itself, and hence the number of roots is constant also, and equal to zero, the value which it has for $x_2 = \cdots = x_n = 0$. The only singularities of f_{x_1}/f inside the circle are those due to the poles of f.

The integral of $x_1 f_{x_1}/f$, multiplied by a constant factor, gives the sum of the values of x_1 for which poles occur, each counted according to its multiplicity, or, as may be said for brevity, the sum of the poles. It is an analytic function of $x_2, \cdots, x_n$, as already stated. By integrating $x_1^2 f_{x_1}/f$, it is shown that the sum of the squares of the poles is analytic, and similarly for the higher powers. It follows that if $p_1(x_2, \cdots, x_n)$, $\cdots$, $p_m(x_2, \cdots, x_n)$, are the elementary symmetric functions of the poles, they are analytic* for all values of $x_2, \cdots, x_n$, satisfying (1), and the polynomial of the mth degree in x_1,

$$\psi(x_1, x_2, \cdots, x_n) = x_1^m - p_1 x_1^{m-1} + \cdots \pm p_m,$$

whose roots correspond in location and multiplicity to the poles of f

---

* Cf. Osgood, Madison Colloquium, 1914, pp. 181–184; Picard, Traité d'analyse, vol. 2, chapter IX, section 8; Goursat, Cours d'analyse mathématique, vol. 2, section 356.

in the neighborhood $|x_1| \leqq r_1$ of the origin, is analytic in all n of its arguments.

The function

$$\varphi(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)\psi(x_1, x_2, \cdots, x_n),$$

if suitably defined at the points where it has removable discontinuities as a function of x_1, is analytic in x_1 throughout the circle, for each set of values of $x_2, \cdots, x_n$. But if x_1 is any interior point of the circle, φ is expressible in the form

$$\varphi(x_1, x_2, \cdots, x_n) = \frac{1}{2\pi i}\int_C \frac{\varphi(t, x_2, \cdots, x_n)}{t - x_1}\,dt,$$

where C denotes the circumference. The integrand here is an analytic function of the $n + 1$ variables $t, x_1, x_2, \cdots, x_n$, together, all along the path of integration, and the integral consequently is an analytic function of the last n of these variables. That is, φ is analytic throughout the neighborhood of the origin, and the relation

$$f = \frac{\varphi}{\psi}$$

gives the desired representation of f. It follows immediately from the definition of φ that it does not vanish in the neighborhood of the origin, since the zeros of ψ are cancelled by the poles of f, and zeros of f do not exist.

Considered by itself, the present theorem can be proved more simply. For as soon as it has been established that f is different from zero throughout the neighborhood of the origin, it can be shown at once that the reciprocal of f has only removable singularities. But the reasoning based on the introduction of the function ψ will be of use in the proof of the later theorems.

2. Singularity of the second kind, two variables.

THEOREM II. *Let $f(x, y)$ be a function of the complex variables x, y, having the following properties in the neighborhood of the origin:*

(*a*) *If y is held fast, $f(x, y)$ is an analytic function of x, except at most for poles.*

(*b*) *The distribution of these poles is continuous at every point except the origin itself.*

(*c*) *Except at the origin and at the points corresponding to poles, f is analytic in x and y together.*

Then f can be represented throughout a sufficiently restricted neighborhood of the origin, except at the singular points specified above, in the form

$$f(x, y) = \frac{\varphi(x, y)}{\psi(x, y)},$$

where φ and ψ are analytic in both variables throughout the neighborhood in question.

Let ρ_1 be a positive number such that $f(x, 0)$ is analytic for $|x| = \rho_1$. Then if ρ_2 is sufficiently small, $f(x, y)$ will be analytic for $|x| = \rho_1$, when $|y| \leqq \rho_2$. We shall denote the regions $|x| \leqq \rho_1$, $|y| \leqq \rho_2$, by R_1 and R_2 respectively. Let y_0 be any particular value of y, distinct from zero, in R_2. Suppose that the poles of $f(x, y_0)$ in R_1 are situated at the points $x', x'', \cdots, x^{(q)}$. It will be possible to associate with each of the points $x^{(j)}$ a pair of positive numbers $r_1^{(j)}$, $r_2^{(j)}$, in such a way that the circles $|x - x^{(j)}| \leqq r_1^{(j)}$ are wholly exterior to each other and interior to R_1, and that the sum of the orders of the poles in any one circle remains constant as long as $|y - y_0|$ does not exceed the corresponding quantity $r_2^{(j)}$; for the distribution of the poles is assumed to be continuous at each of the points in question. The closed region obtained by subtracting from R_1 the interiors of the small circles, contains no pole of $f(x, y_0)$, and will remain free from poles if y is allowed to vary subject to a suitable inequality $|y - y_0| \leqq r_2^{(0)}$. If r_2 is the smallest of the quantities $r_2^{(0)}$, $r_2^{(j)}$, the sum of the orders of the poles in all R_1 will be constant throughout the neighborhood $|y - y_0| \leqq r_2$. As every point of R_2, except the point $y = 0$, has a neighborhood of this sort, an application of the Heine-Borel theorem leads to the conclusion that the sum of the orders of the poles in R_1 has a single constant value m throughout R_2, except at the one point just specified.

A particular value $y = y_0 \neq 0$ in R_2 being designated once more, the circles about the points $x^{(j)}$ may be supposed chosen so small that none of them contains any zero of $f(x, y_0)$ in its interior or on its boundary. Then the quotient f_{x_1}/f is analytic on the boundaries of these circles for $y = y_0$, and will remain so for $|y - y_0| \leqq r_2$, if r_2 is further diminished sufficiently. The reasoning employed in the proof of the first theorem may be applied to each of the circles in turn; the quotient may be integrated around the circumference, with or without a factor x_1^k. It is found that no zeros come in to complicate the representation of the poles, and that the sum of the poles, or of like powers of the poles, is an analytic function of y. For $|y - y_0| \leqq r_2$, the poles in the small circles are all the poles in R_1. The sum of the kth powers of the poles in R_1, where k is any positive integer, may be obtained by adding the corresponding quantities for the various small circles. The sums of powers, and hence the quantities $p_1(y), \cdots, p_m(y)$, the elementary symmetric functions of the poles, are analytic in y for $|y - y_0| \leqq r_2$.

From this it follows that the functions $p_k(y)$ are analytic throughout R_2, except for $y = 0$. But they remain finite even in the neighborhood of this point, for it is apparent from their meaning that their numerical values can not exceed certain readily assignable quantities involving r_1. If they are defined for $y = 0$, without regard to the distribution of the poles of f, by means of their limiting values, they will be analytic in R_2 without exception, and the function

$$\psi(x, y) = x^m - p_1(y)x^{m-1} + \cdots \pm p_m(y)$$

will be analytic in both variables.

The product $\varphi = f\psi$, then, if suitably defined at points of removable discontinuity, is analytic as a function of x throughout R_1, when y has any value different from zero. But it is readily seen that this is true even for $y = 0$, at least if an exception is made of the origin in the x-plane.

The function $\varphi(x, 0)$ is certainly analytic at any point, distinct from the origin, where $f(x, 0)$ has no pole. Let $x_0 \neq 0$, then, be a point where $f(x, 0)$ has a pole of the λth order. It is to be shown that $\psi(x, 0)$ vanishes here λ times. The point x_0 may be surrounded with a circle of arbitrarily small radius, on the circumference of which $\psi(x, 0)$ has no root; and then $\psi(x, y)$ will have no root on the circumference for sufficiently small values of y. By extending the integral of ψ_x/ψ around this circle, it is seen that the number of roots of $\psi(x, y)$ inside is a continuous function of y for $y = 0$. But this is the same as the number of poles of f inside the circle, when $y \neq 0$, and the latter number remains by hypothesis equal to λ when y varies slightly from the value zero. Consequently $\psi(x, 0)$ has just λ roots inside the circle; and as this is true, however small the circle may be, it must be that there is a λ-fold root at x_0.

The function $\varphi(x, y)$ may now be regarded as analytic in x at all points in the neighborhood of the origin, with the single exception of the point $(0, 0)$ itself. That it is analytic in both variables together, except at this point, is shown by means of Cauchy's integral formula, as in the preceding theorem. If $y \neq 0$, the boundary of R_1 may be taken as the path of integration, while if $y = 0$, the path in the case of any particular point x will be a smaller circle surrounding the point and excluding the origin. Being analytic in both x and y except at the isolated point $(0, 0)$, the function φ has at most a removable discontinuity there, and the representation

$$f(x, y) = \frac{\varphi(x, y)}{\psi(x, y)}$$

is of the form desired.

3. Singularity of the second kind, n variables. In the statement of the next theorem, dealing with the space of the n complex variables $x_1, x_2, \cdots, x_n$, reference will be made to a point-set H, the nature of which it will be convenient to describe in advance. It is situated within a certain neighborhood of the origin, to which the whole discussion is restricted; and its points are finite in number in the space of the variables x_1, x_2, if the values of the remaining $n - 2$ coördinates are fixed. To restate the latter requirement at greater length, if the last $n - 2$ coördinates of a point of H are to have given values, only a finite number of determinations of x_2 are possible, and corresponding to any one of these there are only a finite number of values of x_1.

THEOREM III. *Let $f(x_1, x_2, \cdots, x_n)$ have the following properties in the neighborhood of the origin:*

(*a*) *If $x_2, \cdots, x_n$, are held fast, f is an analytic function of x_1, except at most for poles.*

(*b*) *The distribution of the poles is continuous, except at the points of a set H, of the sort described above.*

(*c*) *Except at the points of H and those corresponding to poles, f is analytic in all n variables together.*

Then f can be represented throughout a sufficiently restricted neighborhood of the origin, except at the singular points just specified, in the form

$$f = \frac{\varphi}{\psi},$$

where φ and ψ are analytic in all n variables throughout the neighborhood in question.

A new proof is required only if the set H includes the origin, since otherwise Theorem I can be applied at once. In any case, the points where f fails to be analytic in all n variables for $x_2 = \cdots = x_n = 0$, whether poles of $f(x_1, 0, \cdots, 0)$ or points of H, are finite in number, and there will be no difficulty in assigning a circle $|x_1| = \rho_1$, on which f is analytic when the last $n - 1$ arguments vanish. It will continue to be analytic on this circle, in all n arguments together, for $|x_i| \leqq \rho_2$, $i = 2, \cdots, n$, if ρ_2 is a sufficiently small positive quantity. Let R_1 denote the region $|x_1| \leqq \rho_1$, R_2 the region in the space of the last $n - 1$ variables defined by the inequalities to which they have just been subjected, and H_2 the set of points in R_2 whose coördinates figure as the last $n - 1$ coördinates of points of H.

By reasoning entirely similar to that used in proving the preceding theorem, it is shown that the sum of the orders of the poles of f in R_1 is constant in the neighborhood of every point of R_2 which is not a point of H_2. It follows that this number has one and the same value throughout the whole of R_2, points of H_2 excluded. For if $x_3, \cdots, x_n$ are held fast, it remains constant as x_2 varies, avoiding a finite number of exceptional values; thus it acquires a significance as a function of $x_3, \cdots, x_n$ alone, and it is seen to remain constant for all changes of these variables without exception.

It is shown next that $p_1, \cdots, p_m$, the elementary symmetric functions of the poles of f in R_1, are analytic functions of $x_2, \cdots, x_n$, at all points of R_2 except those which are contained in H_2; the proof given at the corresponding stage in the derivation of Theorem II is applicable here with practically no change. Furthermore, it is obvious from their meaning that these functions remain finite throughout R_2. By abandoning their interpretation with reference to the poles of f, they may be regarded as analytic in x_2, when $x_3, \cdots, x_n$, are held fast, even for the isolated values of x_2 to which points of H_2 correspond. Such a point may be surrounded by a circle in the x_2-plane, on the circumference of which the functions are analytic in all $n - 1$ variables. The argument by means of Cauchy's integral formula, with which the earlier demonstrations were concluded, may be employed here to show that when suitably defined in H_2 the quantities $p_1, \cdots, p_m$, are analytic throughout the whole of R_2. The function

$$\psi = x_1^m - p_1 x_1^{m-1} + \cdots \pm p_m$$

is analytic in $x_1, \cdots, x_n$, throughout the neighborhood of the origin.

Of the product $f\psi$, it can be said at the outset that it is analytic in x_1 throughout R_1, if suitably defined at points of removable discontinuity, for all sets of values of $x_2, \cdots, x_n$ in R_2, except those which belong to H_2. As in the special case with which the second theorem deals, this statement is true even for the exceptional sets of values of $x_2, \cdots, x_n$, if the values of x_1 which give points of H are left out of account. The function $f\psi$ is analytic in all the variables together for all but a finite number of points of R_1, if $x_2, \cdots, x_n$ are given any particular values, and the conditions are suitable for the application of Cauchy's integral formula once more, with the conclusion that $f\psi$ is analytic in all n variables throughout the neighborhood of the origin, except at the points of H. If $x_3, \cdots, x_n$, are held fast, the points of H are isolated in the space of the variables x_1 and x_2; it follows that $f\psi$ is analytic in x_1 (and x_2) at the points of H, if suitably defined there, and a final application of Cauchy's integral formula shows that it is analytic in all the variables. Thus the proof is completed.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.